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Topology of equilateral polygon linkages

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Abstract

We study the topology of moduli spaces of polygons with fixed side lengths in the Euclidean plane and space. We study their smoothness, then determine their Euler numbers.

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1. Introduction

We consider the spaces M_n^p (respectively $M_n'^p$) of equilateral polygon linkages with n ($n \geqslant 3$) vertices in the fixed Euclidean space $\mathbb{R}^p$ ($p \geqslant 2$) modulo the orientation preserving isometry group (respectively the isometry group) of $\mathbb{R}^p$. More precisely, let

$$C_n^p = \{(u_1, \ldots, u_n) \in (\mathbb{R}^p)^n\colon\ |u_i - u_{i+1}| = 1\ (1 \leqslant i \leqslant n)\}, \tag{1}$$

where we shall understand $|u_n - u_{n+1}| = 1$ as $|u_n - u_1| = 1$. Note that $\mathrm{Iso}(\mathbb{R}^p)$ (= the isometry group of $\mathbb{R}^p$, i.e., semidirect product of $\mathbb{R}^p$ with $O(p)$) naturally acts on C_n^p. Hence $\mathrm{Iso}^+(\mathbb{R}^p)$ (= the orientation preserving isometry group of $\mathbb{R}^p$, i.e., semidirect product of $\mathbb{R}^p$ with $SO(p)$) also acts on C_n^p.

We define M_n^p and $M_n'^p$ by

$$M_n^p = C_n^p/\mathrm{Iso}^+(\mathbb{R}^p), \qquad M_n'^p = C_n^p/\mathrm{Iso}(\mathbb{R}^p). \tag{2}$$

Then it is clear that we have an involution τ on M_n^p such that $M_n'^p = M_n^p/\tau$.

If we define $\widetilde{C}_n^p$ by

$$\widetilde{C}_n^p = \{(u_1, \ldots, u_n) \in C_n^p\colon\ u_{n-1} = O,\ u_n = e_1\}, \tag{3}$$

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where O is the origin and e_1 is the first unit vector of $\mathbb{R}^p$, then it is clear that $M_n^p = \widetilde{C}_n^p/SO(p-1)$ and $M_n'^p = \widetilde{C}_n^p/O(p-1)$. Here we regard $SO(p-1)$ (respectively $O(p-1)$) as the subgroup of $SO(p)$ (respectively $O(p)$) consisting of elements which fix e_1.

In the case $p = 2$, we have the following examples (cf. Examples 2.1): $M_3^2 = \{2 \text{ points}\}$ is clear, and it is easy to see that M_4^2 is homeomorphic to

$$\{(x,y) \in \mathbb{R}^2\colon (x+1)^2+y^2=1\} \cup \{(x,y) \in \mathbb{R}^2\colon (x-1)^2+y^2=1\} \cup \{(x,y) \in \mathbb{R}^2\colon x^2+y^2=4\}.$$

Recently Havel [2] proved that M_5^2 is diffeomorphic to Σ_4, i.e., the compact connected and orientable two dimensional manifold of genus 4. (Other proofs of this result are given in [3,4].)

The purposes of this paper are as follows.

(i) We study the smoothness and real dimensions of M_n^2 and $M_n'^2$ for all n, then determine the Euler numbers of M_n^2 and $M_n'^2$.

(ii) We study the same things as in (i) for M_n^3 and $M_n'^3$.

(iii) We prove the stability of M_n^p, here stability means the case of large p for fixed n, by using the Schoenberg's stability theorem about $M_n'^p$ [7].

(iv) We determine the structure of M_n^p and $M_n'^p$ for all p and $n \leqslant 5$ up to homeomorphism or diffeomorphism, except for the case M_5^3 in which we study only the possibility of the homotopy type.

Hereafter to say "x is a singular point of a topological space X" means that there is no open neighbourhood of x in X which is homeomorphic to some Euclidean space. Hence X is not a topological manifold if X has a singular point. "X is a manifold with singular points" means that $X - \{\text{singular point set}\}$ is a smooth manifold. Manifold is to be without boundary except where otherwise stated.

Concerning (i) we shall prove the following

Theorem A. (a) *For an odd n, M_n^2 and $M_n'^2$ are smooth manifolds of real dimensions $n-3$.*

(b-1) *For an even n, M_n^2 is a manifold with singular points such that $(u_1, \ldots, u_{n-2}, O, e_1) \in M_n^2$ is a singular point if and only if all u_i $(1 \leqslant i \leqslant n-2)$ lie on the x-axis, i.e., the line determined by O and e_1. The generic real dimension of M_n^2 is $n-3$.*

As mentioned above, we have an involution τ on M_n^2. If n is even, it is clear that the fixed point set of τ coincides with the singular point set of M_n^2. Then it is natural to expect that $M_n'^2$ is a topological manifold. But unfortunately we have the following

(b-2) *For an odd m, every singular point of M_{2m}^2 is also a singular point of $M_{2m}'^2$.*

The Euler numbers of M_n^2 and $M_n'^2$ are given by

(c-1)

$$\chi(M_n^2) = \begin{cases} (-1)^{m+1}\binom{2m-1}{m}, & n = 2m, \\ (-1)^{m+1}\binom{2m}{m}, & n = 2m+1. \end{cases}$$

(c-2)

$$\chi(M_n^{'2}) = \begin{cases} \binom{2m-1}{m}, & n = 2m \text{ with an odd } m, \\ 0, & n = 2m \text{ with an even } m, \\ (-1)^{m+1}\binom{2m-1}{m}, & n = 2m+1. \end{cases}$$

Here $\binom{a}{b}$ means the binomial coefficient.

Concerning (ii) we shall prove the following

Theorem B. (a) *For an odd n, M_n^3 is a smooth manifold, while $M_n^{'3}$ is a manifold with possibly singular points such that singular points (if they exist) are contained in $M_n^{'2}$.*

(b-1) *For an even n, M_n^3 is a manifold with possibly singular points such that singular points (if they exist) are represented by $(u_1, \ldots, u_{n-2}, O, e_1) \in \widetilde{C}_n^3$ with all u_i $(1 \leqslant i \leqslant n-2)$ lying on the x-axis.*

(b-2) *For an even n, $M_n^{'3}$ is a manifold (only $M_4^{'3}$ has a boundary) with possibly singular points such that singular points (if they exist) are contained in $M_n^{'2}$.*

Finally the generic real dimensions of M_n^3 and $M_n^{'3}$ are independent of parity of n, and equal to $2n-6$.

The Euler numbers of M_n^3 and $M_n^{'3}$ are given by

(c-1)

$$\chi(M_n^3) = \begin{cases} -2^{2m-2} + \binom{2m}{m}, & n = 2m, \\ -2^{2m-1} + (2m+1)\binom{2m-1}{m}, & n = 2m+1. \end{cases}$$

(c-2)

$$\chi(M_n^{'3}) = \begin{cases} -2^{2m-3} + \frac{3}{2}\binom{2m-1}{m}, & n = 2m \text{ with an odd } m, \\ -2^{2m-3} + \binom{2m-1}{m}, & n = 2m \text{ with an even } m, \\ -2^{2m-2} + \left(m + \frac{1+(-1)^{m+1}}{2}\right)\binom{2m-1}{m}, & n = 2m+1. \end{cases}$$

Concerning (iii) we shall prove the following

Theorem C. (a) *M_n^{n-1} is homeomorphic to $S^{n(n-3)/2}$. For $p \geqslant n$, M_n^p is independent of p so that homeomorphic to $D^{n(n-3)/2}$.*

(b) *$M_n^{'n-2}$ is homeomorphic to $S^{n(n-3)/2-1}$. For $p \geqslant n-1$, $M_n^{'p}$ is independent of p so that homeomorphic to $D^{n(n-3)/2}$. Here, S^a means the a dimensional sphere and D^b means the b dimensional disk.*

We note that Theorem C(b) is a result of Schoenberg.

Finally we state the results of (iv). By Theorem C and the information about M_n^p and $M_n'^p$ ($n \leqslant 5$) as stated above, all we need to know is $M_5'^2$ and M_5^3. Concerning this, we shall prove the following

Theorem D. (a) *$M_5'^2$ is diffeomorphic to $\#_5\mathbb{R}P^2$, i.e., the five times connected sum of $\mathbb{R}P^2$.*

(b) *M_5^3 is homotopically equivalent to either one of $\#_5\mathbb{C}P^2$, $\#_4\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, or $\#_3\mathbb{C}P^2\#_2\overline{\mathbb{C}P^2}$.*

This paper is organized as follows. In Section 2, we first give a short proof of the homeomorphism type of M_4^2, and then prove Theorem A(a), (b-1,2), and (c-2) by assuming (c-1). In Section 3, we prove Theorem A(c-1) and then give a corollary of this (cf. Corollary 3.5). In Section 4, we prove Theorem B(a), (b-1,2), and (c-2) by assuming (c-1). As for Theorem B(c-1), a recurrence to calculate $\chi(M_n^3)$ is given in Section 4 (cf. Proposition 4.8), and in Section 5 we solve this recurrence by using purely facts about elementary number theory (cf. Propositions 5.4 and 5.6, which seem interesting themselves). In Section 6, these facts are established. In Section 7, we prove Theorems C and D.

2. Proof of Theorem A (The first half)

This section and the next treat the case $p = 2$. Note that $M_n^2 = \widetilde{C}_n^2$ as $SO(1) = 1$. First we give examples.

Examples 2.1. (a) $M_3^2 = \{2 \text{ points}\}$. (b) M_4^2 is homeomorphic to

$$\{(x,y) \in \mathbb{R}^2\colon (x+1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2\colon (x-1)^2 + y^2 = 1\} \\ \cup \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 = 4\}.$$

(c) M_5^2 is diffeomorphic to Σ_4.

Proof. (a) is trivial and (c) is proved in [2–4]. (b) is easy but we give a short proof. We write the clockwise angle from $\overrightarrow{e_1 O}$ to $\overrightarrow{e_1 u_1}$ by α, and counterclockwise angle from $\overrightarrow{O e_1}$ to $\overrightarrow{O u_2}$ by β. Then

$$u_1 = \begin{pmatrix} 1 - \cos\alpha \\ \sin\alpha \end{pmatrix} \quad \text{and} \quad u_2 = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}.$$

As $|u_1 - u_2| = 1$, we have

$$\sin\frac{\alpha}{2}\cos\frac{\alpha+\beta}{2}\sin\frac{\beta}{2} = 0,$$

which is the desired result. $\square$

Next we shall prove Theorem A(a), (b-1,2), and (c-2) by admitting (c-1). As for Theorem A(a) and (b-1), we first prove the following proposition. Note that Proposition 2.2(b) is weaker than Theorem A(b-1) with respect to the assertion of singular points.

Proposition 2.2. (a) *For an odd n, M_n^2 is a smooth manifold of real dimension $n-3$.*

(b) *For an even n, M_n^2 is a manifold with possibly singular points such that the singular points of M_n^2 (if they exist) are given by $(u_1, \ldots, O, e_1) \in M_n^2$ with all u_i $(1 \leqslant i \leqslant n-2)$ lying on the x-axis. The generic real dimension of M_n^2 is $n-3$.*

Proof. As the action of $\mathrm{Iso}^+(\mathbb{R}^2)$ on $\mathcal{C}_n^2$ is free, we can argue the smoothness of $\mathcal{C}_n^2$ instead of M_n^2. We write $u_i = \binom{x_i}{y_i}$ for $1 \leqslant i \leqslant n$ and define polynomials f_i $(1 \leqslant i \leqslant n)$ with $2n$ variables $(x_1, y_1, \ldots, x_n, y_n)$ by $f_i(x_1, y_1, \ldots, x_n, y_n) = (x_i - x_{i+1})^2 + (y_i - y_{i+1})^2 - 1$. Of course f_n is defined by $f_n(x_1, y_1, \ldots, x_n, y_n) = (x_n - x_1)^2 + (y_n - y_1)^2 - 1$. Then it is clear that

$$\mathcal{C}_n^2 = \{(x_1, y_1, \ldots, x_n, y_n) \in (\mathbb{R}^2)^n \colon f_i(x_1, y_1, \ldots, x_n, y_n) = 0 \ (1 \leqslant i \leqslant n)\}. \quad (4)$$

We compute the rank of $J = J(f_1, \ldots, f_n)(a_1, b_1, \ldots, a_n, b_n)$ (= Jacobian matrix $J(f_1, \ldots, f_n)$ at a point $(a_1, b_1, \ldots, a_n, b_n)$) for each $(a_1, b_1, \ldots, a_n, b_n) \in \mathcal{C}_n^2$. We define $\alpha_i = a_i - a_{i+1}$, $\beta_i = b_i - b_{i+1}$ for $1 \leqslant i \leqslant n$. Then clearly

$$\sum_{i=1}^n \alpha_i = 0, \qquad \sum_{i=1}^n \beta_i = 0$$

and $\alpha_i^2 + \beta_i^2 = 1$.

In J, add 3rd, 5th, …, $(2n-1)$th columns to 1st column, and add 4th, 6th, …, $2n$th columns to 2nd column. Write this new matrix by $\tilde{J}$. Then it is clear that 2nd, 3rd, …, nth rows of $\tilde{J}$ are linearly independent because $\alpha_i^2 + \beta_i^2 = 1$.

Assume $\operatorname{rank} J < n$. Then 1st row of $\tilde{J}$ must be spanned by 2nd, 3rd, …, nth rows. Write this by

$$1\text{st row} = \varepsilon_2(2\text{nd row}) + \cdots + \varepsilon_n(n\text{th row}). \quad (5)$$

As $\alpha_i^2 + \beta_i^2 = 1$, we know by (5) that $\varepsilon_i = \pm 1$ $(2 \leqslant i \leqslant n)$. Hence we have $\alpha_i = \varepsilon_i \alpha_1$, $\beta_i = \varepsilon_i \beta_1$. Now substitute this in

$$\sum_{i=1}^n \alpha_i = 0, \qquad \sum_{i=1}^n \beta_i = 0,$$

we have (as $\alpha_1^2 + \beta_1^2 = 1$),

$$1 + \varepsilon_2 + \cdots + \varepsilon_n = 0. \quad (6)$$

If n is odd, then (6) has no solution with $\varepsilon_i = \pm 1$. Hence 1st, …, nth rows of J are linearly independent. This shows that $\mathcal{C}_n^2$ is a smooth manifold.

If n is even, $\operatorname{rank} J < n$ if and only if $\alpha_i = \varepsilon_i \alpha_1$, $\beta_i = \varepsilon_i \beta_1$ with $\varepsilon_i = \pm 1$ satisfying (6). But this means that all of $\binom{a_1}{b_1}, \ldots, \binom{a_n}{b_n}$ lie on a same line in $\mathbb{R}^2$. In particular if $\binom{a_{n-1}}{b_{n-1}} = O$, $\binom{a_n}{b_n} = e_1$, then we have the desired result.

The fact that $\dim M_n^2 = n-3$ is obvious.

This completes the proof of Proposition 2.2. $\square$

Now in order to complete the proof of Theorem A(a), all we have to prove is the following

Corollary 2.3. *For an odd n, $M_n'^2$ is a smooth manifold of dimension $n-3$.*

This corollary is clear from Proposition 2.2 and the fact that the involution τ on M_n^2 is free for an odd n.

The rest of this section is devoted to the completion of proofs of Theorem A(b-1), (b-2) and (c-2) by assuming (c-1).

Proof of Theorem A(c-2). If an involution τ acts on a space X with fixed point set X^τ and quotient space X/τ, then the following formula is well known (for example, see [1]) $\chi(X) + \chi(X^\tau) = 2\chi(X/\tau)$.

Then $\chi(M_n'^2)$ is computed from $\chi(M_n^2)$ with the fact $(M_n^2)^\tau = \phi$ if n is odd, and

$$(M_n^2)^\tau = \left\{ \binom{2m-1}{m} \text{ points} \right\} \quad \text{if } n = 2m. \qquad \square$$

Finally we shall complete the proof of Theorem A(b-1) and (b-2). By Proposition 2.2, all we have to prove is the following

Proposition 2.4. (a) *In Proposition* 2.2(b), *each* $(u_1, \ldots, u_{n-2}, O, e_1)$ *where all u_i lie on the x-axis is in fact a singular point.*

(b) *For an odd m, each singular point of M_{2m}^2 is also a singular point of $M_{2m}'^2$.*

Proof. We shall prove only (a). (b) can be proved similarly. By defining $z_i = u_i - u_{i-1}$ $(1 \leqslant i \leqslant n)$, and identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual manner, we can write M_n^2 by

$$M_n^2 = \{(z_1, \ldots, z_{n-1}, 1) \in (S^1)^n \colon z_1 + \cdots + z_{n-1} + 1 = 0\}. \tag{7}$$

We must consider the points $(z_1, \ldots, z_{n-1}, 1)$ with $z_i = \pm 1$ for $1 \leqslant i \leqslant n-1$.

Assume a point $(z_1^0, \ldots, z_{n-1}^0, 1) \in M_n^2$ with $z_i^0 = \pm 1$ has an open neighborhood homeomorphic to $\mathbb{R}^{n-3}$. Take any point $(z_1^1, \ldots, z_{n-1}^1, 1)$ with $z_i^1 = \pm 1$. By an appropriate permutation of coordinates of $z_1, \ldots, z_{n-1}$, we have a homeomorphism $f : M_n^2 \to M_n^2$ such that $f(z_1^0, \ldots, z_{n-1}^0, 1) = (z_1^1, \ldots, z_{n-1}^1, 1)$. This means that $(z_1^1, \ldots, z_{n-1}, 1)$ also has an open neighborhood homeomorphic to $\mathbb{R}^{n-3}$. Hence M_n^2 must be a topological manifold.

But we know $\dim M_n^2 = n-3$, which is odd. Hence we must have $\chi(M_n^2) = 0$. This contradicts Theorem A(c-1).

This completes the proof of Proposition 2.4. $\square$

Remark 2.5. Recently Kapovich and Millson [4] proved that $M_{2m}'^2$ is not a topological manifold for integers $m \geqslant 3$. The proof goes as follows: It follows from [4] that M_{2m}^2

has the same singularity as the quadratic cone in $\mathbb{R}^{2m-2}$ given by the quadratic form Q of the signature $(m-1, m-1)$ (since the linkages are equilateral, the number m of "backtracks" in any degenerate polygon is the same as the number of "forward-tracks"). The reflection $(x,y) \mapsto (-x,y)$ on $\mathbb{R}^2$ corresponds to the involution τ on M^2_{2m} so that $M'^2_{2m} = M^2_{2m}/\tau$. The involution τ acts on this quadratic cone as the antipodal map (it has an isolated fixed point P corresponding to a degenerate polygon). In particular, link of each singular point $[P] \in M'^2_{2m}$ is not simply connected unless $m = 2$.

On the other hand, the involution on M^2_4 corresponds to the reflection of the three circles about the x-axis in the notation of Example 2.1(b). Hence M'^2_4 is homeomorphic to S^1.

3. The Euler number of M^2_n

For a real number $l > 0$, we set $\xi(l) = \binom{l}{0} \in \mathbb{R}^2$ and define $M^2_n(l)$ by

$$M^2_n(l) = \{(u_1, \dots, u_n) \in (\mathbb{R}^2)^n\colon\ u_{n-1} = O,\ u_n = \xi(l),\ |u_i - u_{i+1}| = 1\ (1 \leqslant i \leqslant n-2 \text{ or } i = n)\}. \quad (8)$$

Of course $M^2_n(1)$ is M^2_n in our old notation. Note that $M^2_n(n-1) = \{1 \text{ point}\}$ and $M^2_n(l) = \emptyset$ for $l > n-1$.

The calculation of $\chi(M^2_n(1))$ is executed as follows: (i) First we find a formula to compute $\chi(M^2_n(l))$ (here $1 \leqslant l \leqslant n-1$ is a natural number) from $\chi(M^2_{n-1}(l'))$ (here $1 \leqslant l' \leqslant n-2$ is a natural number). (ii) Next we inductively solve this formula. Hence actually we determine $\chi(M^2_n(l))$ for all natural numbers $n \geqslant 3$ and $l \geqslant 1$.

To simplify computations, we use the following fact which can be proved in the same way as the proof of Proposition 2.2.

Proposition 3.1. (a) *For an odd n, $M^2_n(l)$ is a smooth manifold of dimension $n-3$ if l is an odd natural number or $l \notin \mathbb{N}$.*

(b) *For an even n, $M^2_n(l)$ is a smooth manifold of dimension $n-3$ if l is an even natural number or $l \notin \mathbb{N}$.*

Hence $\chi(M^2_n(l)) = 0$ in the case of (b), *because $M^2_n(l)$ is a compact, odd dimensional manifold.*

Now we shall prove the following

Proposition 3.2.
(a)

$$\chi(M^2_n(1)) = \begin{cases} -2\chi(M^2_{n-1}(1)) + \chi(M^2_{n-1}(2)), & n \text{ even}, \\ 2\chi(M^2_{n-1}(1)), & n \text{ odd}. \end{cases}$$

(b) *For a natural number $l \geqslant 2$,*

$$\chi(M_n^2(l)) = \begin{cases} 0, & n \text{ even and } l \text{ even}, \\ \chi(M_{n-1}^2(l-1)) - 2\chi(M_{n-1}^2(l)) + \chi(M_{n-1}^2(l+1)), & \\ \quad n \text{ even and } l \text{ odd}, & \\ \chi(M_{n-1}^2(l-1)) + \chi(M_{n-1}^2(l+1)), & n \text{ odd and } l \text{ even}, \\ 2\chi(M_{n-1}^2(l)), & n \text{ odd and } l \text{ odd}. \end{cases}$$

Of course $M_n^2(l) = \emptyset$ *for* $l > n-1$. *Hence we shall regard* $\chi(M_n^2(l)) = 0$ *in this case.*

Proof. We define $\pi : M_n^2(l) \to S^1$ by $\pi(u_1, \ldots, u_{n-2}, 0, e_1) = u_{n-2}$, and define Y_n by

$$Y_n = \{(u_1, \ldots, u_{n-1}, O) \in (\mathbb{R}^2)^n\colon\ |u_i - u_{i+1}| = 1\ (1 \leqslant i \leqslant n-2),\\ |u_1| = |u_{n-1}| = 1\}. \tag{9}$$

Actually we know $\chi(Y_n)$ by the following

Assertion 3.3. $\chi(Y_n) = 0$.

In fact we have a $S^1 = SO(2)$ action on Y_n by the rotation around O, i.e., $g(u_1, \ldots, u_{n-1}, O) = (gu_1, \ldots, gu_{n-1}, O), g \in S^1$. It is clear that the fixed point set $Y_n^{S^1} = \emptyset$. Hence this assertion follows easily by the well known formula (for example, see [1]) that $\chi(Y_n) = \chi(Y_n^{S^1}) = 0$.

Proof of Proposition 3.2 *(continued)*. Now we treat the case $l = 1$.

(i) The case of an even n: By Proposition 2.2, we know that singular fibers of π are $\pi^{-1}(e_1)$ and $\pi^{-1}(-e_1)$. It is clear that these singular fibers are homeomorphic to Y_{n-2} and $M_{n-1}^2(2)$ respectively. Hence we have a fiber bundle $M_n^2(1) - (Y_{n-2} \sqcup M_{n-1}^2(2)) \to S^1 - (e_1 \sqcup (-e_1))$. But this is equivalent to

$$M_n^2(1) - (Y_{n-2} \sqcup M_{n-1}^2(2)) \cong \mathbb{R} \times M_{n-1}^2(1) \sqcup \mathbb{R} \times M_{n-1}^2(1). \tag{10}$$

Now, as $M_n^2(1) - (Y_{n-2} \sqcup M_{n-1}^2(2))$ is a smooth manifold by Proposition 3.1, Poincaré–Lefschetz duality (modulo 2 coefficients) shows

$$H^*(M_n^2(1), Y_{n-2} \sqcup M_{n-1}^2(2)) \\ \cong H_{n-3-*}(\mathbb{R} \times M_{n-1}^2(1)) \oplus H_{n-3-*}(\mathbb{R} \times M_{n-1}^2(1)).$$

As $n-3$ is odd, we have

$$\chi(M_n^2(1)) = \chi(Y_{n-2}) + \chi(M_{n-1}^2(2)) - 2\chi(M_{n-1}^2(1)).$$

We have the result from this with Assertion 3.3.

(ii) The case of an odd n: In this case, the singular fibers of π are identified with $Y_{n-2} \sqcup M_{n-1}^2(1) \sqcup M_{n-1}^2(1)$. Hence the result follows easily by the same argument as (i) with Proposition 3.1 and Assertion 3.3.

The case $l \geqslant 2$ can be proved similarly. For example if n is even and l is odd, then singular fibers of π are homeomorphic to $M_{n-1}^2(l-1) \sqcup M_{n-1}^2(l+1)$, and we have

$M_n^2(l) - (M_{n-1}^2(l-1) \sqcup M_{n-1}^2(l+1)) \cong \mathbb{R} \times M_{n-1}^2(l) \sqcup \mathbb{R} \times M_{n-1}^2(l)$. Hence by the Poincaré–Lefschetz duality, we will have the result.

This completes the proof of Proposition 3.2. □

It is trivial that $M_3^2(1) = \{2 \text{ points}\}, M_3^2(2) = \{1 \text{ point}\}$, and $M_3^2(l) = \emptyset$ for $l > 2$. Hence $\chi(M_n^2(l))$ ($n \geqslant 3$ and $l \geqslant 1$) will be determined successively from Proposition 3.2 with initial conditions about $\chi(M_3^2(l))$. Actually $\chi(M_n^2(l))$ is given by the following explicit formula.

Theorem 3.4.

$$\chi(M_n^2(l)) = \begin{cases} 0, & n = 2m,\ l = 2k, \\ (-1)^{m+k+1}\binom{2m-1}{m+k}, & n = 2m,\ l = 2k+1, \\ (-1)^{m+k}\frac{k}{m}\binom{2m}{m-k}, & n = 2m+1,\ l = 2k, \\ 2(-1)^{m+k+1}\binom{2m-1}{m+k}, & n = 2m+1,\ l = 2k+1. \end{cases}$$

This theorem can be proved easily by induction on n. Hence we omit the details.

This completes the proof of Theorem A(c-1).

Before we leave this section, we give a corollary of Theorem A(c-1).

Corollary 3.5. *For a prime n, $\chi(M_n^2) \equiv -1 \bmod n$.*

Of course this is an easy exercise of elementary number theory. But we have the following more geometric proof of this.

Another proof of Corollary 3.5. For each n (n need not to be a prime), we can define an action of $\mathbb{Z}_n$ (= the cyclic group of n elements) on M_n^2 in the following manner: Let $\sigma \in \mathbb{Z}_n$ be a generator, and let $(u_1, \ldots, u_{n-2}, O, e_1) \in M_n^2$. Then $\sigma(u_1, \ldots, u_{n_2}, O, e_1)$ is given by "Fixing $(u_1, \ldots, u_{n-2}, O, e_1)$ with wire and move u_1 to u_2, u_2 to u_3, $\ldots, e_1$ to u_1 by an element of $\mathrm{Iso}^+(\mathbb{R}^2)$". Thus, in the notation of (7), we have $\sigma(z_1, z_2, \ldots, z_{n-1}, 1) = (z_{n-1}^{-1}, z_{n-1}^{-1}z_1, \ldots, z_{n-1}^{-1}z_{n-2}, 1)$.

It is easy to show that $(z_1, \ldots, z_{n-1}, 1)$ is a fixed point if and only if it is of the form $(\omega^{n-1}, \omega^{n-2}, \ldots, \omega^2, \omega, 1)$, where $\omega^n = 1, \omega \neq 1$. Hence we see that the fixed point set $(M_n^2)^{\mathbb{Z}_n}$ consists of $n-1$ points. In particular if n is a prime, then we have by the well known formula (for example, see [1]) that $\chi(M_n^2) \equiv \chi((M_n^2)^{\mathbb{Z}_n}) \equiv -1 \bmod n$.

This completes the proof of Corollary 3.5. □

4. Proof of Theorem B (The first half)

First we prove Theorem B(a), (b-1,2), and (c-2) by assuming (c-1).

Proofs of Theorem B(a) and (b-1,2). By the same argument as in the proof of Proposition 2.2, we can show that C_n^3 is a smooth manifold if n is odd, and a manifold with possibly singular points if n is even. Moreover singular points (if they exist) are of the form $(u_1, \ldots, u_n) \in C_n^3$ such that all u_i's lie on the same line. Finally the dimension of C_n^3 is independent of parity of n, and is equal to $2n$.

If n is odd, then the action of $\mathrm{Iso}^+(\mathbb{R}^3)$ on C_n^3 is free. Hence M_n^3 is smooth.

If n is even, then $(u_1, \ldots, u_n) \in C_n^3$ has a nontrivial isotropy subgroup if and only if $(u_1, \ldots, u_n)$ is of the form above. Hence the assertion about M_n^3 follows.

Finally, the assertion about $M_n^{\prime 3}$ is an easy consequence of the result about M_n^3 with the fact $(M_n^3)^\tau = M_n^{\prime 2}$ (cf. Proposition 7.1). For example, the fact that only $M_4^{\prime 3}$ has boundary is a consequence of the fact that $\dim M_n^{\prime 2} < \dim M_n^3 - 1$ for $n > 4$.

This completes the proofs of Theorem B(a) and (b-1,2). □

Proof of Theorem B(c-2). We know that $\chi(M_n^3)+\chi((M_n^3)^\tau) = 2\chi(M_n^{\prime 3})$. As mentioned above, we know that $(M_n^3)^\tau = M_n^{\prime 2}$. Hence $\chi(M_n^{\prime 3})$ will be calculated from Theorem A(c-2) and Theorem B(c-1).

This completes the proof of Theorem B(c-2). □

Remark 4.1. In the assertions (a), (b-1), and (b-2) of Theorem B, one can actually drop the words "(if they exist)" since the existence of singularities for $n \geqslant 6$ was established in [5]. The proof goes essentially in the same way as in Remark 2.5. That is, to get the picture of singularities in M_{2m}^3, extend Q to a hermitian form Q^c in $\mathbb{C}^{2m-2}$ of the signature $(m-1, m-1)$ and take the quotient of the cone $Q^c = 0$ by the following action of the circle $U(1): (z, w) \mapsto (\lambda z, \lambda^{-1} w)$. Here $Q^c(z, 0) > 0$, $Q^c(0, w) < 0$, $z, w \in \mathbb{C}^{m-1}$.

The neighborhood of zero of this quotient is diffeomorphic to the neighborhood of $P \in M_{2m}^3$. To describe the singularity of $M_{2m}^{\prime 3}$, recall that it is the quotient of M_{2m}^3 by the involution which acts on $\mathbb{C}^{2m-2}$ as the complex conjugation.

The rest of this section is devoted to the proof of Theorem B(c-1). The essential idea of proof is the same as Theorem A(c-1), but this time the situation is a little more complicated because we must divide $\widetilde{C}_n^3$ by $SO(2)$.

For a real number $l > 0$, we set

$$\xi(l) = \begin{pmatrix} l \\ 0 \\ 0 \end{pmatrix}.$$

As in Section 3, we define $X_n(l), M_n(l)$, and Y_n as follows.

$$X_n(l) = \big\{(u_1, \ldots, u_n) \in (\mathbb{R}^3)^n:\ u_{n-1} = O,\ u_n = \xi(l),\\ |u_i - u_{i+1}| = 1\ (1 \leqslant i \leqslant n-2 \text{ or } i = n)\big\}, \tag{11}$$

$$M_n(l) = X_n(l)/SO(2), \tag{12}$$

$$Y_n = \big\{(u_1, \ldots, u_{n-1}, O) \in (\mathbb{R}^3)^n:\ |u_i - u_{i+1}| = 1\ (1 \leqslant i \leqslant n-2),\\ |u_1| = |u_{n-1}| = 1\big\}/SO(2). \tag{13}$$

In (12), of course $SO(2)$ acts on $X_n(l)$ as the subgroup of $SO(3)$ consisting of elements which fix $\xi(l)$. Similarly in (13), the action of $SO(2)$ is defined by the rotation around the x-axis.

Remark 4.2. Note that $M_n(1)$ is equal to M_n^3 in our old notation. And $M_n(l)$ corresponds to $M_n^2(l)$ in the case $\mathbb{R}^2$. (If we follow the notation of Section 3, $M_n(l)$ should be denoted by $M_n^3(l)$. But we shall omit "3" to simplify the notation.) We also note that if we define a similar space as $X_n(l)$ in $\mathbb{R}^2$, this coincides with $M_n^2(l)$, because we need not divide by a group in the case $\mathbb{R}^2$. Finally Y_n in (13) corresponds to Y_n in (9). We use the same notation Y_n for (9) and (13). But we remark that these spaces are *not* the same. As we will not use the definition (9) in this section, there is no danger of confusion.

By the same argument as the proof of Theorem B(a) and (b-1,2), we can prove the following

Proposition 4.3. (a) *For an odd n, and $l \in \mathbb{R}$ which is not an even natural number, $X_n(l)$ is a smooth manifold of dimension $2n-5$, and $M_n(l)$ is a smooth manifold of dimension $2n-6$.*

(b) *For an even n, and $l \in \mathbb{R}$ which is not an odd natural number, $X_n(l)$ is a smooth manifold of dimension $2n-5$, and $M_n(l)$ is a smooth manifold of dimension $2n-6$.*

Remark 4.4. If n and l are in the cases of Proposition 4.3(a) or (b), we have $\chi(X_n(l)) = 0$ because $X_n(l)$ is a compact, odd dimensional manifold.

First we determine the Euler number of $X_n(l)$.

Proposition 4.5.

$$\chi(X_n(l)) = \begin{cases} \binom{n-1}{(n-1-l)/2}, & n \text{ is odd and } l \text{ is even, or } n \text{ is even and } l \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. As mentioned above, S^1 acts on $X_n(l)$. The fixed point set $X_n(l)^{S^1}$ is empty except for n is odd and l is even, or n is even and l is odd. In these cases, fixed point set consists of $(u_1, \ldots, u_{n-2}, O, \xi(l))$ with all u_i's lying on the x-axis. Hence the fixed point set consists of

$$\binom{n-1}{(n-1-l)/2}$$

points. As $\chi(X_n(l)) = \chi(X_n(l)^{S^1})$, the result follows. □

Next we give a formula to calculate $\chi(Y_n)$ from $\chi(M_n(1))$ and $\chi(X_n(1))$.

Proposition 4.6. $\chi(Y_n) = 2\chi(M_n(1)) - \chi(X_n(1))$.

Proof. We let $SO(2)$ act on S^2 by the rotation around the x-axis. Then we have a map $p: Y_n \to S^2/SO(2)$, $p(u_1, \dots, u_{n-1}, O) = u_{n-2}$. We identify $S^2/SO(2)$ with

$$\left\{ \zeta(\theta) = \begin{pmatrix} \cos\theta \\ 0 \\ \sin\theta \end{pmatrix} : 0 \leqslant \theta \leqslant \pi \right\}.$$

Assertion 4.7. $p^{-1}(\zeta(\theta)) = X_n(1)$ holds for $\theta \neq 0, \pi$. $p^{-1}(\zeta(\theta)) = M_n(1)$ holds for $\theta = 0$ or π.

This assertion is trivial because $\zeta(\theta) \in S^2$ is moved to another point of S^2 by a rotation around the x-axis if and only if $0 < \theta < \pi$.

Proof of Proposition 4.6 (*continued*). Let $U_1 = p^{-1}\{\zeta(\theta)\colon 0 \leqslant \theta \leqslant \pi/2\}$ and $U_2 = p^{-1}\{\zeta(\theta)\colon \pi/2 \leqslant \theta \leqslant \pi\}$. By Assertion 4.7, we know that U_1 and U_2 are homotopically equivalent to $M_n(1)$, and $U_1 \cap U_2 = X_n(1)$. Hence Mayer–Vietoris argument shows Proposition 4.6. □

Finally we shall find a formula which corresponds to Proposition 3.2.

Proposition 4.8.
(a) $\chi(M_n(1)) = \chi(Y_{n-2}) + \chi(X_{n-1}(1)) + \chi(M_{n-1}(2))$.
(b) *For a natural number* $l \geqslant 2$,

$$\chi\big(M_n(l)\big) = \chi\big(M_{n-1}(l-1)\big) + \chi\big(X_{n-1}(l)\big) + \chi\big(M_{n-1}(l+1)\big).$$

Of course $M_n(l) = \emptyset$ for $l > n$, hence we shall regard $\chi(M_n(l)) = 0$ in this case.

Proof. We define $q: M_n(l) \to S^2/SO(2)$ by $q(u_1, \dots, u_{n-2}, O, \xi(l)) = u_{n-2}$, and we use the same notation as in the proof of Proposition 4.6 concerning the identification of $S^2/SO(2)$.

First we prove (a). We know that singular fibers of q are $q^{-1}(\zeta(0)), q^{-1}(\zeta(\pi))$, and $q^{-1}(\zeta(\theta_0))$, where $\zeta(\theta_0)$ satisfies $|\zeta(\theta_0) - e_1| = 1$. And it is easy to show that $q^{-1}(\zeta(0)) = Y_{n-2}, q^{-1}(\zeta(\pi)) = M_{n-1}(2)$, $q^{-1}(\zeta(\theta_0)) = X_{n-1}(1)$. Set

$$E_1 = q^{-1}\big\{\zeta(\theta)\colon 0 < \theta < \theta_0\big\}, \qquad E_2 = q^{-1}\big\{\zeta(\theta)\colon \theta_0 < \theta < \pi\big\}.$$

Assertion 4.9. $\chi(E_1) = 0$, $\chi(E_2) = 0$.

In fact, we know by Proposition 4.3 that the restrictions of q to E_1 and E_2 are fiber bundles. And we know that the Euler numbers of the fibers of these bundles vanish by Remark 4.4. Hence this assertion follows.

Proof of Proposition 4.8 (*continued*). Now Proposition 4.8(a) follows from the Poincaré–Lefschetz duality of the pair $(M_n(1),\ Y_{n-2} \sqcup M_{n-1}(2) \sqcup X_{n-1}(1))$ with Assertion 4.9.

The proof of (b) is similar by using the fact that singular fibers of q are identified with $M_{n-1}(l-1), M_{n-1}(l+1), X_{n-1}(l)$.

This completes the proof of Proposition 4.8. $\square$

It is clear that $M_3(1) = M_3(2) = \{1 \text{ point}\}$, and $M_3(l) = \emptyset$ for $l > 2$. Hence we have

$$\chi\big(M_3(1)\big) = 1, \qquad \chi\big(M_3(2)\big) = 1, \qquad \chi\big(M_3(l)\big) = 0 \quad (l > 2). \tag{14}$$

Moreover we shall see in Section 7 (or by the same argument as in the proof of Proposition 4.8) that $M_4(1)$ is homeomorphic to S^2. Hence we can determine $\chi(M_4(l))$ $(l \geqslant 2)$ and $\chi(Y_4)$ by Propositions 4.5, 4.6, 4.8 as follows:

$$\begin{aligned} &\chi\big(M_4(1)\big) = 2, \qquad \chi\big(M_4(2)\big) = 2, \qquad \chi\big(M_4(3)\big) = 1, \\ &\chi\big(M_4(l)\big) = 0 \quad (l > 3), \qquad \chi(Y_4) = 1. \end{aligned} \tag{15}$$

If we repeat this process, $\chi(M_n(l))$ will be determined successively. In the next two sections, we shall give the explicit formula for $\chi(M_n(1))$.

5. The Euler number of M_n^3

First we simplify the recurrence of Proposition 4.8 in the following manner.

Proposition 5.1.

(a-i) $$\chi(M_{2m}(1)) = 3\chi(M_{2m-2}(1)) + \chi(M_{2m-2}(3)) - \binom{2m-3}{m-2} \quad (m \geqslant 3).$$

(a-ii) $$\begin{aligned} \chi(M_{2m}(2k+1)) = {} & \chi(M_{2m-2}(2k-1)) + 2\chi(M_{2m-2}(2k+1)) \\ & + \chi(M_{2m-2}(2k+3)) \quad (m \geqslant 3,\ k \geqslant 1). \end{aligned}$$

(b-i) $$\chi(M_{2m-1}(1)) = 2\chi(M_{2m-3}(1)) + \chi(M_{2m-2}(2)) + \binom{2m-3}{m-2} \quad (m \geqslant 3).$$

(b-ii) $$\begin{aligned} \chi(M_{2m}(2)) = {} & \chi(M_{2m-1}(1)) + \chi(M_{2m-2}(2)) \\ & + \chi(M_{2m-2}(4)) + \frac{(2m-3)!(3m-4)}{m!(m-2)!} \quad (m \geqslant 3). \end{aligned}$$

(b-iii) $$\begin{aligned} \chi(M_{2m}(2k)) = {} & \chi(M_{2m-2}(2k-2)) + 2\chi(M_{2m-2}(2k)) \\ & + \chi(M_{2m-2}(2k+2)) + \frac{(2m-3)!(4m-4)}{(m-k-1)!(m+k-1)!} \end{aligned}$$

$(m \geqslant 3,\ k \geqslant 2)$.

Note that (a-i,ii) with (15) completely determine $\chi(M_{2m}(l))$ for all $m \geqslant 2$ and an odd $l \geqslant 1$. Note also that (b-i,ii,iii) with (14), (15) completely determine $\chi(M_{2m-1}(1))$ and $\chi(M_{2m}(l))$ for all $m \geqslant 2$ and an even $l \geqslant 2$.

Proposition 5.1 is an easy consequence of repeated use of Proposition 4.8 with Propositions 4.5 and 4.6. Hence we shall omit the details.

First we shall determine $\chi(M_{2m}(l))$ for an odd $l \geqslant 1$. We define a series of polynomials with variable x indexed by odd integers $\geqslant 3$, denoted by

$$\{f_3(x), f_5(x), \ldots, f_{2k+1}(x), \ldots\} \quad (k \geqslant 1),$$

as follows.

Set $f_3(x) = 3x - 1$ and $f_5(x) = 4x^2 - 2x + 4$. Then define $f_{2k+1}(x)$ $(k \geqslant 3)$ inductively by

$$\begin{aligned} f_{2k+1}(x) &= (4x+2)f_{2k-1}(x+1) - (x+k)(x+k-1)f_{2k-3}(x) \\ &\quad - 2(x+k)f_{2k-1}(x). \end{aligned} \tag{16}$$

Fix $2m$ $(m \geqslant 2)$ and $2k+1$ $(k \geqslant 0)$. (We are writing an odd integer l by $l = 2k+1$.) Define a real number indexed by $2m$ and $2k+1$, denoted by $\omega_{2m}(2k+1)$, by

$$\omega_{2m}(1) = -2^{2m-2} + \binom{2m}{m} \quad (m \geqslant 2). \tag{17}$$

$$\omega_{2m}(2k+1) = -2^{2m-2} + \frac{(2m-1)!}{(m+k)!(m-1)!} f_{2k+1}(m) \quad (m \geqslant 2 \text{ and } k \geqslant 1). \tag{18}$$

Then actually we have the following

Theorem 5.2. $\chi(M_{2m}(2k+1)) = \omega_{2m}(2k+1)$ *for* $m \geqslant 2$ *and* $k \geqslant 0$.

Proof. We shall prove by induction on m. The nontrivial part is to check the initial conditions, i.e., to check the following

Proposition 5.3. $\chi(M_4(2k+1)) = \omega_4(2k+1)$ *for* $k \geqslant 0$.

Proof of Theorem 5.2 (*continued*). Assume Proposition 5.3 for the moment. Then the proof of Theorem 5.2 will be completed, because to prove $\chi(M_{2(M+1)}(2k+1)) = \omega_{2(M+1)}(2k+1)$, by assuming $\chi(M_{2m}(2k'+1)) = \omega_{2m}(2k'+1)$ for $m \leqslant M$ and all $k' \geqslant 0$, is an easy computation with Proposition 5.1(a-i,ii) and the definition of $\omega_{2m}(2k+1)$.

Thus the proof of Theorem 5.2 will be completed if we prove Proposition 5.3. □

Proof of Proposition 5.3. $\chi(M_4(1)) = \omega_4(1) = 2$ is trivial. Hence all we have to do is the case $k \geqslant 1$. But we see, by (15) and (17), that it is equivalent to the following

Proposition 5.4. $f_3(2) = 5$ *and* $f_{2k+1}(2) = \frac{2}{3}(k+2)!$ $(k \geqslant 2)$.

We shall prove this proposition in the next section.

Next we shall determine $\chi(M_{2m-1}(1))$ and $\chi(M_{2m}(l))$ for $m \geqslant 2$ and an even $l \geqslant 2$. In the same way as above, we define $\{f_2(x), f_4(x), \ldots, f_{2k}(x), \ldots\}$ $(k \geqslant 1)$ as follows.

Set $f_2(x) = x$ and $f_4(x) = x^2 - x + 2$. Then define $f_{2k}(x)$ $(k \geqslant 3)$ inductively by

$$f_{2k}(x) = (4x+2)f_{2k-2}(x+1) - (x+k-1)(x+k-2)f_{2k-4}(x) \\ - 2(x+k-1)f_{2k-2}(x) - 4\prod_{i=2}^{k}(x-k+i). \tag{19}$$

Fix $2m$ ($m \geqslant 2$) and $2k$ ($k \geqslant 1$). (We are writing an even integer l by $l = 2k$.) Define a real number indexed by $2m$ and $2k$, denoted $\omega_{2m}(2k)$, by

$$\omega_{2m}(2k) = -2^{2m-2} + \frac{(2m-1)!}{(m+k-1)!(m-1)!}f_{2k}(m) \quad (m \geqslant 2 \text{ and } k \geqslant 1). \tag{20}$$

Fix also $2m-1$ ($m \geqslant 2$). Define a real number indexed by $2m-1$, denoted by θ_{2m-1}, by

$$\theta_{2m-1} = -2^{2m-3} + (2m-1)\binom{2m-3}{m-1}. \tag{21}$$

Then actually we have

Theorem 5.5. *$\chi(M_{2m-1}(1)) = \theta_{2m-1}$ for $m \geqslant 2$, and $\chi(M_{2m}(2k)) = \omega_{2m}(2k)$ for $m \geqslant 2$ and $k \geqslant 1$.*

Proof. As in the proof of Theorem 5.2, the only nontrivial part is to check the initial conditions, for which it suffices to prove the following

Proposition 5.6. *$f_2(2) = 2$ and $f_{2k}(2) = \frac{2}{3}(k+1)!$ ($k \geqslant 2$).*

We also prove this proposition in the next section.

To summarize this section, we have proved Theorem B(c-1), assuming the truth of Propositions 5.4 and 5.6.

6. Proofs of Propositions 5.4 and 5.6

In this section we give proofs of Proposition of 5.4 and 5.6 using purely facts about elementary number theory that seem to be interesting themselves.

First we prove Proposition 5.4. In order to prove Proposition 5.4, we must actually prove the following stronger assertion.

Proposition 6.1. *For $k \geqslant 1$ and $0 \leqslant i \leqslant k$, the followings hold.*

$$f_{2k+1}(0) = \begin{cases} -(k!), & k \text{ odd}, \\ 2(k!), & k \text{ even}, \end{cases} \tag{22}$$

$$f_{2k+1}(i) = \frac{2^{2i-2}(i-1)!}{(2i-1)!}(k+i)! \quad (1 \leqslant i \leqslant k). \tag{23}$$

Note that Proposition 6.1 determines $f_{2k+1}(x)$ itself because $\deg f_{2k+1}(x) = k$.

Hereafter we shall prove this by induction on k. Clearly Proposition 6.1 holds for $k = 1$ and 2.

Fix $k \geqslant 2$ and assume that Proposition 6.1 holds for $f_{2k'+1}(x)$ with $k' \leqslant k$. We must know $f_{2k+3}(i)$ for $0 \leqslant i \leqslant k+1$. But $f_{2k+3}(i)$ for $0 \leqslant i \leqslant k-1$ can be computed easily by induction hypothesis with (16).

In order to know $f_{2k+3}(k)$ and $f_{2k+3}(k+1)$, we must know $f_{2k-1}(k), f_{2k-1}(k+1), f_{2k+1}(k+1)$, and $f_{2k+1}(k+2)$. But these are calculated if we know $f_{2k-1}(x)$ and $f_{2k+1}(x)$ themselves. In fact this is possible by Lagrange's interpolation (for example, see [9]). And $f_{2k+1}(x)$ is given by

$$\begin{aligned} f_{2k+1}(x) = {} & \frac{(-1)^k \alpha_k}{k!} \prod_{j=1}^{k} (x-j) \\ & + \sum_{i=1}^{k} \left\{ \frac{(-1)^{k+i} 2^{2i-2} (i-1)!(k+i)!}{i!(2i-1)!(k-i)!} \prod_{j=0,\ j \neq i}^{k} (x-j) \right\}, \end{aligned} \tag{24}$$

where

$$\alpha_k = \begin{cases} -(k!), & k \text{ odd}, \\ 2(k!), & k \text{ even}. \end{cases}$$

Now $f_{2k+3}(k)$ will be calculated by Lemma 6.2(a), and $f_{2k+3}(k+1)$ will be calculated by Lemma 6.2(b).

Lemma 6.2.

(a) $$(k+1) \sum_{i=1}^{k} \frac{(-1)^{k+i} 2^{2i-2}}{i} \binom{k+i}{2i-1} = \begin{cases} 2^{2k}, & k \text{ odd}, \\ 2^{2k}-1, & k \text{ even}. \end{cases}$$

(b) $$(2k+3)(k+2) \sum_{i=1}^{k} \frac{(-1)^{k+i} 2^{2i-1}}{k-i+2} \binom{k+i}{2i} = \begin{cases} 2^{2k+2}-1, & k \text{ odd}, \\ 2^{2k+2}-2k-4, & k \text{ even}. \end{cases}$$

Proof of Lemma 6.2. We prove only (a). (b) can be proved similarly. Write the left hand side of (a) by S. Define $\mu(z)$ by

$$\mu(z) = \sum_{j=1}^{k} \frac{(-1)^{j+1}}{2^{2j}(k-j+1)} \frac{(1+z)^{2k-j+1}}{z^j}. \tag{25}$$

It is clear that $S = (k+1)2^{2k} \cdot [\text{constant term of } \mu(z)]$, and it is easy to show that

$$\mu(z) = \frac{1}{2^{2k+2}} \frac{(1+z)^k \int_0^{4z(1+z)} (u^k - 1)/(u+1)\, \mathrm{d}u}{z^{k+1}}. \tag{26}$$

Here "constant term of $\mu(z)$" means the a_0 when $\mu(z)$ is expanded into the Laurent series

$$\mu(z) = \sum_{i=-\infty}^{\infty} a_i z^i.$$

It is elementary that

$$\left[\text{constant term of } \frac{(1+z)^k}{z^{k+1}} \int_0^{4z(1+z)} \frac{u^k}{u+1}\, \mathrm{d}u \right] = \frac{2^{2k+2}}{k+1},$$

and

$$\left[\text{constant term of } \frac{(1+z)^k}{z^{k+1}} \int_0^{4z(1+z)} \frac{1}{u+1}\,du\right] = \begin{cases} 0, & k \text{ odd}, \\ 4/(k+1), & k \text{ even}. \end{cases}$$

Hence (a) follows.

This completes the proof of Lemma 6.2, and consequently of Proposition 5.4. □

Proposition 5.6 will be proved in the same way as above, or in the following way which deduces Proposition 5.6 to Proposition 5.4.

For $k \geqslant 2$, define $g_{2k}(x)$ by

$$g_{2k}(x) = f_{2k}(x) - (x-k-1)\prod_{i=1}^{k-1}(x-i).$$

Then it is easy to prove that $g_4(x) = f_3(x), g_6(x) = f_5(x)$, and

$$\begin{aligned} g_{2k}(x) = {} & (4x+2)g_{2k-2}(x+1) - (x+k-1)(x+k-2)g_{2k-4}(x) \\ & - 2(x+k-1)g_{2k-2}(x). \end{aligned}$$

Thus we have $g_{2k}(x) = f_{2k-1}(x)$ $(k \geqslant 2)$. Hence Proposition 5.6 follows.

This completes the proof of Propositions 5.4 and 5.6, and consequently of Theorem B(c-1). □

7. Proofs of Theorems C and D

Before giving the proofs of Theorems C and D, we make some remarks on the connection between M_n^p and M_n^{p+1}, or $M_n^{\prime p}$ and $M_n^{\prime p+1}$. We consider $\mathbb{R}^p$ as a subspace of $\mathbb{R}^{p+1}$ consisting of elements whose $(p+1)$-th coordinates are 0. Let $i^p : \mathbb{R}^p \to \mathbb{R}^{p+1}$ be the inclusion. Naturally i^p induces maps $I_n^p : M_n^p \to M_n^{p+1}$, and $I_n^{\prime p} : M_n^{\prime p} \to M_n^{\prime p+1}$. With this in mind, we shall give the following

Proposition 7.1. (a) *$I_n^{\prime p}$ is an embedding for all n and p. $M_n^{\prime p} = M_n^{\prime p+1}$ for $p \geqslant n-1$, hence $M_n^{\prime p} = M_n^{\prime n-1}$ for $p \geqslant n-1$.*

(b-1) *For $p \geqslant n$, the involution τ on M_n^p is trivial. Hence $M_n^p = M_n^{\prime n-1}$ for $p \geqslant n$.*

(b-2) *While, I_n^p is not an embedding for $p < n$, and in this case $(M_n^p)^\tau = M_n^{\prime p-1}$ holds.*

(c) *In general, M_n^p and $M_n^{\prime p}$ are manifolds with possibly boundary and singular points.*

Proof. For example, the fact $M_n^{\prime p} = M_n^{\prime p+1}$ $(p \geqslant n-1)$ follows from the fact that for each $(u_1, \ldots, u_{n-2}, O, e_1) \in \widetilde{C}_n^p$, the subspace of $\mathbb{R}^p$ spanned by $\{u_1, \ldots, u_{n-2}, e_1\}$ has dimension at most $n-1$.

Other assertions are also easy, so we shall omit the details. □

The following theorem by Schoenberg [7] describes the topology of $M_n^{\prime n-1}$, which is the stable topology of $M_n^{\prime p}$ (cf. Proposition 7.1), and $M_n^{\prime n-2}$.

Theorem 7.2. (a) *$M_n'^{n-1}$ is homeomorphic to $D^{n(n-3)/2}$.*

(b) *Under the homeomorphism of* (a), *$M_n'^{n-2}$ corresponds to the boundary of disk. Hence $M_n'^{n-2}$ is homeomorphic to $S^{n(n-3)/2-1}$.*

Of course Theorem 7.2 is the assertion of Theorem C(b).

Proof of Theorem C(a). The assertion about M_n^p $(p \geqslant n)$ follows easily from Theorem 7.2 with Proposition 7.1.

About M_n^{n-1}, we know $(M_n^{n-1})^\tau = M_n'^{n-2} = S^{n(n-3)/2-1}$, and $M_n^{n-1}/\tau = M_n'^{n-1} = D^{n(n-3)/2}$. Hence M_n^{n-1} will be reproduced if we take two copies of $D^{n(n-3)/2}$ and attach with their boundaries. Of course this is $S^{n(n-3)/2}$. Hence the assertion about M_n^{n-1} follows.

This completes the proof of Theorem C. □

Next we shall prove Theorem D.

Proof of Theorem D(a). It is easy to see that the involution τ on M_5^2 is free. Hence the quotient map $M_5^2 \to M_5'^2$ is a double covering space. As $M_5^2 = \Sigma_4$, we have $\chi(M_5'^2) = -3$, hence $M_5'^2 = \#_5\mathbb{R}P^2$.

This completes the proof of Theorem D(a). □

Finally we consider M_5^3. We know that M_5^3 is a smooth manifold of dimension 4 with an involution τ, such that $(M_5^3)^\tau$ is diffeomorphic to $\#_5\mathbb{R}P^2$, M_5^3/τ is homeomorphic to S^4. Hence it is natural to expect that M_5^3 is diffeomorphic to either one of $\#_5\mathbb{C}P^2$, $\#_4\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, or $\#_3\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$ on which the involution acts by complex conjugation. But in this paper, we content ourselves by studying only the possibility of the homotopy type.

Proof of Theorem D(b). It is not difficult to show, by a general position argument, that M_5^3 is simply connected. We know by Theorem B that $\chi(M_5^3) = 7$, hence $H_2(M_5^3;\mathbb{Z}) = \mathbb{Z}^5$. From this, we have that M_5^3 is of type I, i.e., $x \cdot x$ is odd for some $x \in H_2(M_5^3;\mathbb{Z})$, because if type II then the signature of M_5^3 must be divided by 8 [6]. Hence [6, Corollary 2] shows that M_5^3 is homotopically equivalent to either one of $\#_5\mathbb{C}P^2$, $\#_4\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, or $\#_3\mathbb{C}P^2 \#_2 \overline{\mathbb{C}P^2}$.

This completes the proof of Theorem D(b). □

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